

A New Necessary Condition for the Hyponormality of Toeplitz Operators on the Bergman Space

Željko Čučković

Department of Mathematics, University of Toledo, Toledo, Ohio 43606

Raúl E. Curto

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242

Abstract

A well known result of C. Cowen states that, for a symbol $\varphi \in L^\infty$, $\varphi \equiv \bar{f} + g$ ($f, g \in H^2$), the Toeplitz operator T_φ acting on the Hardy space of the unit circle is hyponormal if and only if $f = c + T_{\bar{h}}g$, for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$. In this note we consider possible versions of this result in the *Bergman* space case. Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+$, $m < n$ and $p < q$. By letting T_φ act on vectors of the form

$$z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

we study the asymptotic behavior of a suitable matrix of inner products, as $k \rightarrow \infty$. As a result, we obtain a sharp inequality involving the above mentioned data:

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|.$$

This inequality improves a number of existing results, and it is intended to be a precursor of basic necessary conditions for joint hyponormality of tuples of Toeplitz operators acting on Bergman spaces in one or several complex variables.

Email addresses: zeljko.cuckovic@utoledo.edu (Željko Čučković),
rcurto@math.uiowa.edu (Raúl E. Curto)

URL: <http://math.utoledo.edu/~zcuckov/> (Željko Čučković),
<http://www.math.uiowa.edu/~rcurto/> (Raúl E. Curto)

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1. Notation and Preliminaries

A bounded operator acting on a complex, separable, infinite dimensional Hilbert space \mathcal{H} is said to be normal if $T^*T = TT^*$; quasinormal if T commutes with T^*T ; subnormal if $T = N|_{\mathcal{H}}$, where N is normal on a Hilbert space \mathcal{K} which contains \mathcal{H} and $N\mathcal{H} \subseteq \mathcal{H}$; hyponormal if $T^*T \geq TT^*$; and 2-hyponormal if (T, T^2) is (jointly) hyponormal, that is

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] \\ [T^*, T^2] & [T^{*2}, T^2] \end{pmatrix} \geq 0.$$

Clearly,

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{2-hyponormal} \Rightarrow \text{hyponormal}.$$

In this paper we focus primarily on the cases $H^2(\mathbb{T})$ and $A^2(\mathbb{D})$, the Hardy space on the unit circle \mathbb{T} and the Bergman space on the unit disk \mathbb{D} , respectively. For these Hilbert spaces, we look at Toeplitz operators, that is, the operators obtained by compressing multiplication operators on the respective L^2 spaces to the above mentioned Hilbert spaces. We consider possible versions, in the Bergman space context, of C. Cowen's characterization of hyponormality for Toeplitz operators on Hardy space of the unit circle. Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+$, $m < n$ and $p < q$. By letting T_φ act on vectors of the form

$$z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

we study the asymptotic behavior of a suitable matrix of inner products, as $k \rightarrow \infty$. As a result, we obtain a sharp inequality involving the above mentioned data. We begin with a brief survey of the known results in the Hardy space context.

2. The Hardy Space Case

Let $L^2(\mathbb{T})$ denote the space of square integrable functions with respect to the Lebesgue measure on the unit circle, and let $H^2(\mathbb{T})$ denote the subspace consisting of functions with vanishing negative Fourier coefficients; equivalently, $H^2(\mathbb{T})$ is the $L^2(\mathbb{T})$ -closure of the space of analytic polynomials. We also

let $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ denote the corresponding Banach spaces of essentially bounded functions on \mathbb{T} . The orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ will be denoted by P .

Given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with *symbol* φ is $T_\varphi : H^2(\mathbb{T}) \rightarrow H^2(\mathbb{T})$, given by $T_\varphi f := P(\varphi f)$ ($f \in H^2(\mathbb{T})$). T_φ is said to be *analytic* if $\varphi \in H^\infty(\mathbb{T})$.

P.R. Halmos's Problem 5 ([9]) asks whether every subnormal Toeplitz operator is either normal or analytic. In 1984, C. Cowen and J. Long answered this question in the negative [4]. Along the way, C. Cowen obtained a characterization of hyponormality for Toeplitz operators, as follows [3]: if $\varphi \in L^\infty$, $\varphi = \bar{f} + g$ ($f, g \in H^2$), then T_φ is hyponormal $\Leftrightarrow f = c + T_h g$, for some $c \in \mathbb{C}$, $h \in H^\infty$, and $\|h\|_\infty \leq 1$. T. Nakazi and K. Takahashi [17] later found an alternative description: For $\varphi \in L^\infty$, let $\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty\}$; then T_φ is hyponormal $\Leftrightarrow \mathcal{E}(\varphi) \neq \emptyset$. (For a generalization of Cowen's result, see [8].) In this note we take a first step toward finding suitable generalizations of these results to the case of Toeplitz operators on the Bergman space over the unit disk. We also wish to pursue appropriate generalizations of the results on joint hyponormality of pairs of Toeplitz operators on the Hardy space, obtained in [6] and [5].

At present, there is no known characterization of subnormality of Toeplitz operators in the unit circle in terms of the symbol. However, we do know that every 2-hyponormal Toeplitz operator with a *trigonometric* symbol is subnormal [6]. Thus, a suitable intermediate goal is to find a characterization of 2-hyponormality in terms of the symbol, perhaps using as a starting point either Cowen's or Nakazi-Takahashi's characterizations of hyponormality.

For Toeplitz operators with trigonometric symbols, the following results describe hyponormality.

Proposition 2.1. [20] *Suppose*

$$\varphi(z) \equiv \sum_{k=0}^n a_k z^k + \overline{\sum_{k=0}^n b_k z^k},$$

with $a_n \neq 0$. Let

$$\begin{pmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \vdots \\ \bar{c}_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Then T_φ is hyponormal if and only if $|\Phi_k(c_0, \dots, c_k)| \leq 1$ ($0 \leq k \leq n-1$), where Φ_k denotes the Schur function introduced in [19].

Proposition 2.2. *Let φ be a trigonometric polynomial of the form*

$$\varphi(z) = \sum_{n=-m}^N a_n z^n.$$

- (i) (D. Farenick and W.Y. Lee [7]) If T_φ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.
- (ii) (D. Farenick and W.Y. Lee [7]) If T_φ is a hyponormal operator then $N - m \leq \text{rank}[T_\varphi^*, T_\varphi] \leq N$.
- (iii) (R. Curto and W.Y. Lee [6]) The hyponormality of T_φ is independent of the particular values of the Fourier coefficients a_0, a_1, \dots, a_{N-m} of φ . Moreover, for T_φ hyponormal, the rank of the self-commutator of T_φ is independent of those coefficients.
- (iv) (D. Farenick and W.Y. Lee [7]) If $m \leq N$ and $|a_{-m}| = |a_N| \neq 0$, then T_φ is hyponormal if and only if the following equation holds:

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \vdots \\ \bar{a}_N \end{pmatrix}. \quad (\text{hyponormal.})$$

In this case, the rank of $[T_\varphi^*, T_\varphi]$ is $N - m$.

- (v) (D. Farenick and W.Y. Lee [7]) T_φ is normal if and only if $m = N$, $|a_{-N}| = |a_N|$, and (hyponormal.) holds with $m = N$.

3. The Bergman Space Case

By analogy with the case of the unit circle, let $L^\infty \equiv L^\infty(\mathbb{D})$, $H^\infty \equiv H^\infty(\mathbb{D})$, $L^2 \equiv L^2(\mathbb{D})$ and $A^2 \equiv A^2(\mathbb{D})$ denote the relevant spaces in the case of the unit disk \mathbb{D} . Similarly, let $P : L^2 \rightarrow A^2$ denote the orthogonal projection onto the Bergman space. For $\varphi \in L^\infty$, the Toeplitz operator on the Bergman space with symbol φ is

$$T_\varphi : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D}),$$

given by

$$T_\varphi f := P(\varphi f) \quad (f \in A^2).$$

T_φ is said to be *analytic* if $\varphi \in H^\infty$.

3.1. A Revealing Example

Let

$$\varphi \equiv \bar{z}^2 + 2z.$$

On the Hardy space $H^2(\mathbb{T})$, T_φ is not hyponormal, because $m = 2$, $N = 1$, and $m > N$ (see Proposition 2.2).

However, on the Bergman space $A^2(\mathbb{D})$ T_φ is hyponormal, as we now prove. Consider a slight variation of the symbol, that is,

$$\varphi \equiv \bar{z}^2 + \alpha z. \quad (\alpha \in \mathbb{C})$$

Observe that

$$\langle [T_\varphi^*, T_\varphi]f, f \rangle = \langle |\alpha|^2 [T_{\bar{z}}, T_z] + [T_{z^2}, T_{\bar{z}^2}]f, f \rangle$$

so that T_φ is hyponormal if and only if

$$|\alpha|^2 \|zf\|^2 + \langle P(\bar{z}^2 f), \bar{z}^2 f \rangle \geq |\alpha|^2 \langle P(\bar{z} f), \bar{z} f \rangle + \|z^2 f\|^2 \quad (3.1)$$

for all $f \in A^2(\mathbb{D})$.

A calculation now shows that this happens precisely when $|\alpha| \geq 2$, as follows. For, given $f \in A^2(\mathbb{D})$, $f \equiv \sum_0^\infty b_n z^n$, one can apply Lemma 3.1 and obtain

$$\|zf\|^2 = \sum_0^\infty \frac{|b_n|^2}{n+2},$$

$$\|P(\bar{z}f)\|^2 = \sum_1^\infty |b_n|^2 \frac{n}{(n+1)^2},$$

$$\|z^2 f\|^2 = \sum_0^\infty \frac{|b_n|^2}{n+3},$$

and

$$\|P(\bar{z}^2 f)\|^2 = \sum_2^\infty |b_n|^2 \frac{n-1}{(n+1)^2}.$$

Thus, (3.1) becomes

$$|\alpha|^2 \sum_0^\infty \frac{|b_n|^2}{n+2} + \sum_2^\infty |b_n|^2 \frac{n-1}{(n+1)^2} \geq |\alpha|^2 \sum_1^\infty |b_n|^2 \frac{n}{(n+1)^2} + \sum_0^\infty \frac{|b_n|^2}{n+3}. \quad (3.2)$$

In short, equation (3.2) must hold for every sequence (b_n) of coefficients of f . Consider first a sequence (b_n) with $b_0 := 1$ and $b_n := 0$ for all $n \geq 1$. By (3.1), we have $|\alpha|^2 \geq \frac{2}{3}$. Next, take $b_0 := 0$, $b_1 := 1$ and $b_n := 0$ for all $n \geq 2$; then (3.1) yields $|\alpha|^2 \geq 3$. Finally, if we fix $k \geq 2$ and we use a sequence (b_n) defined as $b_0 := 0$, $b_1 := 0$, \dots , $b_{k-1} := 0$, $b_k := 1$ and $b_n := 0$ for all $n > k$, then (3.2) becomes

$$\frac{|\alpha|^2}{k+2} + \frac{k-1}{(k+1)^2} \geq |\alpha|^2 \frac{k}{(k+1)^2} + \frac{1}{k+3}.$$

This immediately leads to the condition

$$|\alpha|^2 \geq 4 \cdot \frac{k+2}{k+3} \quad (\text{all } k \geq 2);$$

that is, $|\alpha|^2 \geq 4$. As a result, T_φ is hyponormal if and only if $|\alpha| \geq 2$. It follows that $T_{\bar{z}^2+2z}$ is hyponormal.

3.2. A Key Difference Between the Hardy and Bergman Cases

Lemma 3.1. For $u, v \geq 0$, we have

$$P(\bar{z}^u z^v) = \begin{cases} 0 & v < u \\ \frac{(v-u+1)}{v+1} z^{v-u} & v \geq u \end{cases}.$$

PROOF.

$$\begin{aligned} P(\bar{z}^u z^v) &= \sum_{j=0}^{\infty} \left\langle \bar{z}^u z^v, \frac{z^j}{\|z^j\|} \right\rangle \frac{z^j}{\|z^j\|} \\ &= \sum_{j=0}^{\infty} \frac{\langle \bar{z}^u z^v, z^j \rangle z^j}{\|z^j\|^2} = \sum_{j=0}^{\infty} (j+1) \langle z^v, z^{u+j} \rangle z^j \\ &= \begin{cases} 0 & v < u \\ \frac{v-u+1}{v+1} z^{v-u} & v \geq u \end{cases}. \end{aligned}$$

□

Corollary 3.3. For $v \geq u$ and $t \geq w$, we have

$$\begin{aligned} \langle P(\bar{z}^u z^v), P(\bar{z}^w z^t) \rangle &= \left\langle \frac{v-u+1}{v+1} z^{v-u}, \frac{t-w+1}{t+1} z^{t-w} \right\rangle \\ &= \frac{(t-w+1)}{(v+1)(t+1)} \delta_{u+t, v+w}. \end{aligned}$$

3.3. Some Known Results

In this subsection, we briefly summarize a number of partial results relating to the Bergman space case.

- (H. Sadraoui [18]) If $\varphi \equiv \bar{g} + f$, the following are equivalent:
 - (i) T_φ is hyponormal on $A^2(\mathbb{D})$;
 - (ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_f^* H_f$;
 - (iii) $H_{\bar{g}} = C H_f$, where C is a contraction on $A^2(\mathbb{D})$.
- (I.S. Hwang [10]) Let $\varphi \equiv a_{-m} \bar{z}^m + a_{-N} \bar{z}^N + a_m z^m + a_N z^N$ ($0 < m < N$) satisfying $a_m \bar{a}_N = a_{-m} a_{-N}$, then T_φ is hyponormal if and only if

$$\begin{aligned} \frac{1}{N+1} (|a_N|^2 - |a_{-N}|^2) &\geq \frac{1}{m+1} (|a_{-m}|^2 - |a_m|^2) \quad (\text{if } |a_{-N}| \leq |a_N|) \\ N^2 (|a_{-N}|^2 - |a_N|^2) &\leq m^2 (|a_m|^2 - |a_{-m}|^2) \quad (\text{if } |a_N| \leq |a_{-N}|). \end{aligned}$$

The last condition is not sufficient.

- (I.S. Hwang [11]) Let $\varphi \equiv 4\bar{z}^3 + 2\bar{z}^2 + \bar{z} + z + 2z^2 + \beta z^3$ ($|\beta| = 4$). Then T_φ is hyponormal if and only if $\beta = 4$.

- (I.S. Hwang [11]) Let

$$\varphi \equiv 8\bar{z}^3 + \bar{z}^2 + \beta\bar{z} + \gamma z + 7z^2 + 2z^3 \quad (|\beta| = |\gamma|).$$

Then T_φ is not hyponormal.

- (P. Ahern and Ž. Čučković [1]) Let $\varphi \equiv \bar{g} + f \in L^\infty(\mathbb{D})$, and assume that T_φ is hyponormal. Then

$$Bu \geq u,$$

where B denotes the Berezin transform and $u := |f|^2 - |g|^2$.

- (P. Ahern and Ž. Čučković [1]) Let $\varphi \equiv \bar{g} + f \in L^\infty(\mathbb{D})$, and assume that T_φ is hyponormal. Then

$$\lim_{z \rightarrow \zeta} (|f'(z)|^2 - |g'(z)|^2) \geq 0$$

for all $\zeta \in \mathbb{T}$. In particular, if f' and g' are continuous at $\zeta \in \mathbb{T}$ then $|f'(\zeta)| \geq |g'(\zeta)|$.

- (I.S. Hwang and J. Lee [14]) The authors obtain some basic results on hyponormality of Toeplitz operators on weighted Bergman spaces.
- (Y. Lu and C. Liu [15]) The authors obtain necessary and sufficient conditions for the hyponormality of T_φ in the case when φ is a radial symbol.
- (Y. Lu and Y. Shi [16]) The authors study the weighted Bergman space case, and prove the following result.

Theorem 3.4. (cf. [16, Theorem 2.4(ii)]) Let $\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^m + \delta \bar{z}^n$, with $n > m$. Then $m^2(|\beta|^2 - |\gamma|^2) + n^2(|\alpha|^2 - |\delta|^2) \geq mn|\bar{\alpha}\beta - \bar{\gamma}\delta|$.

3.4. Hyponormality of Toeplitz Operators on the Bergman Space

The self-commutator of T_φ is

$$C := [T_\varphi^*, T_\varphi]$$

We seek necessary and sufficient conditions on the symbol φ to ensure that $C \geq 0$.

The following result gives a flavor of the type of calculations we face when trying to decipher the hyponormality of a Toeplitz operator acting on the Bergman space. Although the calculation therein will be superseded by the calculations in the following section, it serves both as a preliminary example and as motivation for the organization of our work.

Proposition 3.5. Assume $k, \ell \geq \max\{a, b\}$. Then

$$\begin{aligned} & \langle [T_{\bar{z}^a}, T_{z^b}] (z^k + cz^\ell), z^k + cz^\ell \rangle \\ = & a^2 \left[\frac{1}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{1}{(\ell+1)^2(\ell+1+a)} \right] \delta_{a,b} \\ & + ac \left[\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)} \delta_{a+k,b+\ell} \right. \\ & \left. + \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)} \delta_{a+\ell,b+k} \right] \end{aligned}$$

PROOF. Keeping in mind that $k, \ell \geq \max\{a, b\}$, we calculate the action of the commutator on the binomial $z^k + cz^\ell$.

$$\begin{aligned} & \langle [T_{\bar{z}^a}, T_{z^b}] (z^k + cz^\ell), z^k + cz^\ell \rangle \\ = & \langle T_{\bar{z}^a} T_{z^b} (z^k + cz^\ell), z^k + cz^\ell \rangle - \langle T_{z^b} T_{\bar{z}^a} (z^k + cz^\ell), z^k + cz^\ell \rangle \\ = & \langle z^{b+k} + cz^{b+\ell}, z^{a+k} + cz^{a+\ell} \rangle - \langle P(\bar{z}^a z^k + c\bar{z}^a z^\ell), P(\bar{z}^b z^k + c\bar{z}^b z^\ell) \rangle \\ = & \frac{\delta_{a,b}}{a+k+1} + c \frac{\delta_{a+k,b+\ell}}{a+k+1} + c \frac{\delta_{a+\ell,b+k}}{a+\ell+1} + c^2 \frac{\delta_{a,b}}{a+\ell+1} \\ & - \frac{(k-b+1)}{(k+1)^2} \delta_{a,b} - c \frac{(k-a+1)}{(k+1)(\ell+1)} \delta_{a+\ell,b+k} \\ & - c \frac{(k-b+1)}{(k+1)(\ell+1)} \delta_{a+k,b+\ell} - c^2 \frac{(\ell-b+1)}{(\ell+1)^2} \delta_{a,b} \\ = & \left[\frac{1}{a+k+1} + c^2 \cdot \frac{1}{a+\ell+1} - \frac{(k-b+1)}{(k+1)^2} - c^2 \cdot \frac{(\ell-b+1)}{(\ell+1)^2} \right] \delta_{a,b} \\ & + c \left[\frac{1}{a+k+1} - \frac{(k-b+1)}{(k+1)(\ell+1)} \right] \delta_{a+k,b+\ell} \\ & + c \left[\frac{1}{a+\ell+1} - \frac{(k-a+1)}{(k+1)(\ell+1)} \right] \delta_{a+\ell,b+k} \\ = & \left[\frac{(k+1)(b-a) + ab}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{(\ell+1)(b-a) + ab}{(\ell+1)^2(\ell+1+a)} \right] \delta_{a,b} \\ & + c \cdot \left[\frac{1}{a+k+1} - \frac{(\ell-a+1)}{(k+1)(\ell+1)} \right] \delta_{a+k,b+\ell} \\ & + c \left[\frac{1}{a+\ell+1} - \frac{(k-a+1)}{(k+1)(\ell+1)} \right] \delta_{a+\ell,b+k} \\ = & \left[\frac{a^2}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{a^2}{(\ell+1)^2(\ell+1+a)} \right] \delta_{a,b} \\ & \text{(if } a \neq b \text{ then the whole expression is 0)} \\ & + ac \cdot \frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)} \delta_{a+k,b+\ell} \\ & + ac \cdot \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)} \delta_{a+\ell,b+k} \end{aligned}$$

$$\begin{aligned}
&= a^2 \left[\frac{1}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{1}{(\ell+1)^2(\ell+1+a)} \right] \delta_{a,b} \\
&\quad + ac \left[\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)} \delta_{a+k,b+\ell} \right. \\
&\quad \left. + \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)} \delta_{a+\ell,b+k} \right],
\end{aligned}$$

as desired. \square

Corollary 3.7. *Assume $a = b$, $k, \ell \geq a$ and $k \neq \ell$. Then*

$$\begin{aligned}
&\langle [T_{\bar{z}^a}, T_{z^a}] (z^k + cz^\ell), z^k + cz^\ell \rangle \\
&= a^2 \left[\frac{1}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{1}{(\ell+1)^2(\ell+1+a)} \right].
\end{aligned}$$

3.5. Self-Commutators

We focus on the action of the self-commutator C of certain Toeplitz operators T_φ on suitable vectors f in the space $A^2(\mathbb{D})$. The symbol φ and the vector f are of the form

$$\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q \quad (n > m; p < q)$$

and

$$f := z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

respectively, with ℓ and r to be determined later. We also assume that $n - m = q - p$. Our ultimate goal is to study the asymptotic behavior of this action as k goes to infinity. Thus, we consider the expression $\langle Cf, f \rangle$, given by

$$\langle [(T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q})^*, T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q}] (z^k + cz^\ell + dz^r), z^k + cz^\ell + dz^r \rangle,$$

for large values of k (and consequently large values of ℓ and r .) It is straightforward to see that $\langle Cf, f \rangle$ is a quadratic form in c and d , that is,

$$\langle Cf, f \rangle \equiv A_{00} + 2 \operatorname{Re}(A_{10}c) + 2 \operatorname{Re}(A_{01}d) + A_{20}c\bar{c} + 2 \operatorname{Re}(A_{11}\bar{c}d) + A_{02}d\bar{d}. \quad (3.3)$$

Alternatively, the matricial form of (3.3) is

$$\left\langle \begin{pmatrix} A_{00} & A_{10} & A_{01} \\ \bar{A}_{10} & A_{20} & A_{11} \\ \bar{A}_{01} & \bar{A}_{11} & A_{02} \end{pmatrix} \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \right\rangle. \quad (3.4)$$

We now observe that the coefficient A_{00} arises from the action of C on the monomial z^k , that is,

$$A_{00} = \langle Cz^k, z^k \rangle \equiv \langle [(T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q})^*, T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q}] z^k, z^k \rangle.$$

Similarly,

$$\begin{aligned} A_{10} &= \langle Cz^\ell, z^k \rangle, \\ A_{01} &= \langle Cz^r, z^k \rangle, \end{aligned} \quad (3.5)$$

$$\begin{aligned} A_{20} &= \langle Cz^\ell, z^\ell \rangle, \\ A_{11} &= \langle Cz^r, z^\ell \rangle, \\ A_{02} &= \langle Cz^r, z^r \rangle. \end{aligned} \quad (3.6)$$

To calculate A_{00} explicitly, we first recall that the algebra of Toeplitz operators with analytic symbols is commutative, and therefore T_{z^n} commutes with T_{z^m} , T_{z^p} and T_{z^q} .

We also recall that two monomials z^u and z^v are orthogonal whenever $u \neq v$. As a result, the only nonzero contributions to A_{00} must come from the inner products $\langle [T_{z^n}^*, T_{z^n}]z^k, z^k \rangle$, $\langle [T_{z^m}^*, T_{z^m}]z^k, z^k \rangle$, $\langle [T_{z^p}^*, T_{z^p}]z^k, z^k \rangle$ and $\langle [T_{z^q}^*, T_{z^q}]z^k, z^k \rangle$.

Applying Corollary 3.7 we see that

$$A_{00} = \frac{1}{(k+1)^2} \left(\frac{|\alpha|^2 n^2}{k+n+1} + \frac{|\beta|^2 m^2}{k+m+1} - \frac{|\gamma|^2 p^2}{k+p+1} - \frac{|\delta|^2 q^2}{k+q+1} \right).$$

Similarly,

$$\begin{aligned} A_{10} &= \bar{\alpha}\beta \left(\frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)} \right) \delta_{n+k, m+\ell} \\ &\quad + \alpha\bar{\beta} \left(\frac{1}{\ell+n+1} - \frac{k-n+1}{(k+1)(\ell+1)} \right) \delta_{m+k, n+\ell} \\ &\quad - \bar{\gamma}\delta \left(\frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)} \right) \delta_{q+k, p+\ell} \\ &\quad - \gamma\bar{\delta} \left(\frac{1}{\ell+q+1} - \frac{k-q+1}{(k+1)(\ell+1)} \right) \delta_{p+k, q+\ell}. \end{aligned} \quad (3.7)$$

Now recall that $m < n$ and $k < \ell$, so that $m+k < n+\ell$, and therefore $\delta_{m+k, n+\ell} = 0$. Also, $p < q$ implies $p+k < q+\ell$, so that $\delta_{p+k, q+\ell} = 0$. As a consequence,

$$\begin{aligned} A_{10} &= \bar{\alpha}\beta \left(\frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)} \right) \delta_{n+k, m+\ell} \\ &\quad - \bar{\gamma}\delta \left(\frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)} \right) \delta_{q+k, p+\ell}. \end{aligned} \quad (3.8)$$

Consider now A_{01} , as described in (3.5). We wish to imitate the calculation for A_{10} . Observe that $k < r$, so that the vanishing of the relevant δ 's in (3.7) still holds. Thus, we obtain

$$\begin{aligned} A_{01} &= \bar{\alpha}\beta \left(\frac{1}{r+m+1} - \frac{k-m+1}{(k+1)(r+1)} \right) \delta_{n+k, m+r} \\ &\quad - \bar{\gamma}\delta \left(\frac{1}{r+p+1} - \frac{k-p+1}{(k+1)(r+1)} \right) \delta_{q+k, p+r}. \end{aligned} \quad (3.9)$$

In short, A_{01} can be obtained from A_{10} by replacing ℓ by r . In a completely analogous way, we can calculate A_{11} , by replacing ℓ by r and k by ℓ in (3.7):

$$\begin{aligned} A_{11} &= \bar{\alpha}\beta\left(\frac{1}{r+m+1} - \frac{\ell-m+1}{(\ell+1)(r+1)}\right)\delta_{n+\ell,m+r} \\ &\quad - \bar{\gamma}\delta\left(\frac{1}{r+p+1} - \frac{\ell-p+1}{(r+1)(\ell+1)}\right)\delta_{q+\ell,p+r}. \end{aligned} \quad (3.10)$$

Also, A_{20} and A_{02} follow the pattern of A_{00} :

$$A_{20} = \frac{1}{(\ell+1)^2} \left(\frac{|\alpha|^2 n^2}{\ell+n+1} + \frac{|\beta|^2 m^2}{\ell+m+1} - \frac{|\gamma|^2 p^2}{\ell+p+1} - \frac{|\delta|^2 q^2}{\ell+q+1} \right),$$

and

$$A_{02} = \frac{1}{(r+1)^2} \left(\frac{|\alpha|^2 n^2}{r+n+1} + \frac{|\beta|^2 m^2}{r+m+1} - \frac{|\gamma|^2 p^2}{r+p+1} - \frac{|\delta|^2 q^2}{r+q+1} \right).$$

Recall again that $k < \ell < r$. We now make a judicious choice to simplify the forms of A_{10} , A_{11} and A_{01} . That is, we let $\ell := n+k-m$ and $r := \ell+q-p$. It follows that $n+k = m+\ell < m+r$ and $q+k < q+\ell = p+r$. Therefore, both Kronecker deltas appearing in A_{01} are zero, and thus $A_{01} = 0$. Moreover,

$$A_{10} = \bar{\alpha}\beta\left(\frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)}\right) \quad (3.11)$$

$$- \bar{\gamma}\delta\left(\frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)}\right)\delta_{q+k,p+\ell}. \quad (3.12)$$

and

$$\begin{aligned} A_{11} &= \bar{\alpha}\beta\left(\frac{1}{r+m+1} - \frac{\ell-m+1}{(r+1)(\ell+1)}\right)\delta_{n+\ell,m+r} \\ &\quad - \bar{\gamma}\delta\left(\frac{1}{r+p+1} - \frac{\ell-p+1}{(r+1)(\ell+1)}\right). \end{aligned}$$

The 3×3 matrix associated with C becomes

$$M := \begin{pmatrix} A_{00} & A_{10} & 0 \\ \bar{A}_{10} & A_{20} & A_{11} \\ 0 & \bar{A}_{11} & A_{02} \end{pmatrix}.$$

We now wish to study the asymptotic behavior of $k^3 M$ as $k \rightarrow \infty$. Surprisingly, $k^3 A_{00}$, $k^3 A_{02}$ and $k^3 A_{20}$ all have the same limit as $k \rightarrow \infty$. Also, $k^3 A_{10}$ and $k^3 \bar{A}_{11}$ have the same limit. To see this, observe that

$$k^3 A_{00} = \frac{k^2}{(k+1)^2} \left(\frac{k|\alpha|^2 n^2}{k+n+1} + \frac{k|\beta|^2 m^2}{k+m+1} - \frac{k|\gamma|^2 p^2}{k+p+1} - \frac{k|\delta|^2 q^2}{k+q+1} \right),$$

so that

$$a := \lim_{k \rightarrow \infty} k^3 A_{00} = |\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2.$$

Then $\lim_{k \rightarrow \infty} k^3 A_{20} = \lim_{k \rightarrow \infty} k^3 A_{02} = a$. In terms of the remaining entries of $k^3 M$, recall the assumption $n - m = q - p$, and let $g := n - m = q - p$. It follows that $\ell = k + g$ and $r = \ell + g = k + 2g$. By using these values in (3.11), we obtain

$$k^3 A_{10} = \bar{\alpha} \beta \frac{k^3 mn}{(k+1)(k+g+1)(k+g+m+1)} - \bar{\gamma} \delta \frac{k^3 pq}{(k+1)(k+g+1)(k+g+p+1)},$$

so that

$$\rho := \lim_{k \rightarrow \infty} k^3 A_{10} = \bar{\alpha} \beta mn - \bar{\gamma} \delta pq.$$

The calculation for A_{11} is entirely similar, and one gets $\lim_{k \rightarrow \infty} k^3 A_{11} = \rho$.

It follows that the asymptotic behavior of $k^3 M$ is given by the tridiagonal matrix

$$\begin{pmatrix} a & \rho & 0 \\ \bar{\rho} & a & \rho \\ 0 & \bar{\rho} & a \end{pmatrix}.$$

Now, if instead of using a vector of the form

$$f := z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

with $\ell = k + g$ and $r = \ell + g = k + 2g$ (that is, a vector of the form

$$f := z^k + cz^{k+g} + dz^{k+2g}$$

we were to use a longer vector with similar power structure,

$$f_N := z^k + c_1 z^{k+g} + c_2 z^{k+2g} + \dots + c_N z^{k+Ng},$$

it is not hard to see that the asymptotic behavior of the associated matrix would still be given by the tridiagonal matrix with a in the diagonal and ρ in the super-diagonal. To see this, one only need to observe that the entries of the matrix P associated with $\langle Cf, f \rangle$ will follow the same pattern as the entries in M . For example, when $N = 3$ the $(3, 4)$ -entry of P will follow the pattern of A_{10} above, with ℓ and k replaced by $k + 3g$ and $k + 2g$, resp. Similarly, the $(2, 4)$ -entry of P will follow the pattern of A_{01} above, with r and k replaced by $k + 3g$ and $k + g$, resp. As a result, it is straightforward to see that, like A_{01} , the entry P_{24} will be zero. As for P_{34} , one gets

$$\begin{aligned} P_{34} &= \bar{\alpha} \beta \left(\frac{1}{k+3g+m+1} - \frac{k+2g-m+1}{(k+2g+1)(k+3g+1)} \right) \delta_{n+k+2g, m+k+3g} \\ &\quad - \bar{\gamma} \delta \left(\frac{1}{k+3g+p+1} - \frac{k+2g-p+1}{(k+3g+1)(k+2g+1)} \right) \delta_{q+k+2g, p+k+3g}. \end{aligned} \tag{3.13}$$

As before,

$$\begin{aligned} k^3 P_{34} &= \bar{\alpha}\beta \frac{k^3 mn}{(k+2g+1)(k+3g+1)(k+3g+m+1)} \\ &\quad - \bar{\gamma}\delta \frac{k^3 pq}{(k+2g+1)(k+3g+1)(k+3g+p+1)}, \end{aligned}$$

and once again,

$$\lim_{k \rightarrow \infty} k^3 P_{34} = \bar{\alpha}\beta mn - \bar{\gamma}\delta pq = \rho.$$

In summary, the hyponormality of T_φ , detected by the positivity of the self-commutator C , leads to the positive semi-definiteness of the tridiagonal $(N+1) \times (N+1)$ matrix P . Since this must be true for all $N \geq 1$, it follows that a necessary condition for the hyponormality of T_φ is the positive semi-definiteness of the infinite tridiagonal matrix

$$Q := \begin{pmatrix} a & \rho & 0 & \cdots \\ \bar{\rho} & a & \rho & \cdots \\ 0 & \bar{\rho} & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now consider the spectral behavior of Q as an operator on $\ell^2(\mathbb{Z}_+)$.

Lemma 3.8. *For $a \in \mathbb{R}$ and $\rho \in \mathbb{C}$, the spectrum of the infinite tridiagonal matrix Q is $[a - 2|\rho|, a + 2|\rho|]$.*

PROOF. This result is well known; we present a proof for the sake of completeness. Observe that Q is the canonical matrix representation of the Toeplitz operator on $H^2(\mathbb{T})$ with symbol $\varphi(z) := a + 2\operatorname{Re}(\bar{\rho}z)$. Since the symbol is harmonic, it follows that the spectrum of $T_\varphi \equiv aI + T_{\bar{\rho}z + \rho\bar{z}}$ is the set $a + 2\operatorname{Re}(\{\bar{\rho}z : z \in \mathbb{D}\}^-) = a + 2[-|\rho|, |\rho|]$, as desired. \square

As a consequence, if Q is positive (as an operator on $\ell^2(\mathbb{Z}_+)$), then

$$a \geq 2|\rho|.$$

3.6. Main Result

Theorem 3.10. *Assume that T_φ is hyponormal on $A^2(\mathbb{D})$, with*

$$\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q \quad (n > m; p < q).$$

Assume also that $n - m = q - p$. Then

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2|\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|. \quad (3.14)$$

3.7. A Specific Case

When $p = m$ and $q = n$ in

$$\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q \quad (n > m; p < q).$$

the inequality

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|.$$

reduces to

$$n^2(|\alpha|^2 - |\delta|^2) + m^2(|\beta|^2 - |\gamma|^2) \geq 2mn |\bar{\alpha}\beta - \bar{\gamma}\delta|.$$

This not only generalizes previous estimates, but also sharpens them, since previous results did not include the factor 2 in the right-hand side.

4. When is T_φ normal?

We conclude this paper with a description of those symbols φ in Theorem 3.10 which produce a normal operator T_φ . We first recall a result of S. Axler and Ž. Čučković.

Lemma 4.1. ([2]) *Let φ be harmonic and bounded on \mathbb{D} . Then T_φ is normal if and only if there exist a pair of complex numbers a and b such that $(a, b) \neq (0, 0)$ and $F := a\varphi + b\bar{\varphi}$ is constant on \mathbb{D} .*

Assume now that T_φ is normal. By Lemma 4.1, there exist a and b such that $(a, b) \neq (0, 0)$ and $F := a\varphi + b\bar{\varphi}$ is constant. In what follows, we write a harmonic symbol as $\varphi \equiv f + \bar{g}$, with f and g analytic. A straightforward calculation using $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, applied to F , shows that $(|a|^2 - |b|^2) \frac{\partial f}{\partial \bar{z}} = 0$. If f is constant, a similar calculation shows that g is also constant, and a fortiori φ is constant. Thus, without loss of generality, we can assume that f is not constant, and therefore $|a| = |b| > 0$. If we write $a = |a| e^{i\theta}$ and $b = |a| e^{i\eta}$, it is not hard to see that $\varphi + e^{i(\eta-\theta)} \bar{\varphi}$ is constant on \mathbb{D} . Let $\lambda := e^{i(\eta-\theta)}$, so that $|\lambda| = 1$. We conclude that $\varphi + \lambda \bar{\varphi}$ is constant on \mathbb{D} .

Theorem 4.2. *Let $\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q$, with $n < m, p < q, n-m = q-p$ and $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$. Then T_φ is normal if and only if there exists $\lambda \in \mathbb{T}$ such that φ is of one of exactly three types:*

- (i) $\varphi = \alpha z^n - \lambda \bar{\alpha} \bar{z}^n$ (when $n = p$);
- (ii) $\varphi = \alpha z^n + \beta z^m - \lambda(\bar{\alpha} \bar{z}^n + \bar{\beta} \bar{z}^m)$ (when $m = p$); or
- (iii) $\varphi = \beta z^m - \lambda \bar{\beta} \bar{z}^m$ (when $m = q$).

PROOF. (\implies) Assume that T_φ is normal. From the discussion in the paragraph immediately preceding Theorem 4.2, we can always assume that $\varphi + \lambda \bar{\varphi}$ is constant on \mathbb{D} , for some $\lambda \in \mathbb{T}$. Since φ is clearly nonconstant, we know that $G := \varphi + \lambda \bar{\varphi}$ is a constant trigonometric polynomial, with analytic monomials z^m, z^n, z^p and z^q . Since φ is a nonconstant harmonic function, in the above

mentioned list of four monomials we must necessarily have at least two identical monomials. Since $m < n$, $p < q$ and $n - m = q - p$, we are led to consider the following three cases:

(i) $n = p$ (and therefore $m < n = p < q$); here

$$G = \beta z^m + (\alpha + \lambda \bar{\gamma})z^n + \lambda \bar{\delta} \bar{z}^q + \overline{\lambda \beta z^m + (\alpha + \lambda \bar{\gamma})z^n + \lambda \bar{\delta} \bar{z}^q},$$

from which it easily follows that $\beta = 0$, $\gamma = -\lambda \bar{\alpha}$ and $\delta = 0$. Then $\varphi = \alpha z^n - \lambda \bar{\alpha} \bar{z}^n$, as desired.

(ii) $m = p$ (and therefore $m = p < q = n$); here

$$G = (\alpha + \lambda \bar{\delta})z^n + (\beta + \lambda \bar{\gamma})z^m + \overline{\lambda(\alpha + \lambda \bar{\delta})z^n + (\beta + \lambda \bar{\gamma})z^m},$$

so that $\alpha + \lambda \bar{\delta} = 0$ and $\beta + \lambda \bar{\gamma} = 0$. It readily follows that $\delta = -\lambda \bar{\alpha}$ and $\gamma = -\lambda \bar{\beta}$. We then get $\varphi = \alpha z^n + \beta z^m - \lambda \bar{\alpha} \bar{z}^n - \lambda \bar{\beta} \bar{z}^m$, as desired.

(iii) $m = q$, which leads to $\varphi = \beta z^m - \lambda \bar{\beta} \bar{z}^m$.

(\Leftarrow) For the converse, observe that in each of the three representations we have $\varphi + \lambda \bar{\varphi} = 0$, which implies $T_\varphi^* = -\lambda T_\varphi$. Therefore, T_φ^* commutes with T_φ , so T_φ is normal.

The proof is now complete. \square

Remark 4.4. *The form of (i), (ii) and (iii) in Theorem 4.2 is entirely consistent with Theorem 3.10. For instance, consider case (i); here $\beta = \delta = 0$ and $\gamma = -\lambda \bar{\alpha}$, so that both sides of (3.14) are equal to 0.*

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